HYPERSONIC BOUNDARY-LAYER FLOW WITH MASS TRANSFER ON POWER-LAW BODIES

G. N. Dudin

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1. The need to investigate flow with mass transfer arises from the great influence of such flow on the aerodynamic characteristics of a vehicle, and on heat transfer with the environment. For example, forced blowing is an effective method of reducing convective and radiative heat transfer to an exposed surface. Mass transfer can substantially alter the effective shape of a body, and influence both boundary layer separation and the formation of secondary flows.

The investigation of mass transfer has been the subject of a large number of experimental and theoretical studies. Reference [1], e.g., has reviewed investigation of the influence of forced blowing and suction on the characteristics of the two-dimensional steady boundary layer on a permeable surface. An especially relevant aspect at present is investigation of the influence of mass transfer on the three-dimensional flow of a viscous gas at hypersonic flight speeds [2, 3].

The present paper examines symmetric flow of a hypersonic viscous gas over a slender power-law body with mass transfer. The coordinate system is rectangular (Fig. 1). The x^0 axis is aligned along the velocity vector of the incident flow U_∞. The quantities u^0, v^0, w^0 are components of the velocity vector in the boundary layer, along the axes x^0, y^0, z^0, respectively. On the body surface $v^0 = F^0(x^0, z^0)$. The body shape is given by the equation $y^0 = \delta_w^0(x^0, z^0)$. Taking into account that the flow considered is that over a slender body, we can introduce variables [4], fixed on the body surface, $y_* = y^0 - \delta_w^0(x^0, z^0)$, $v_* = v^0 - u^0\partial\delta_w^0/\partial x^0 - w^0\partial\delta_w^0/\partial z^0$. In accordance with the usual estimate for the boundary layer in hypersonic flow [5], we introduce the dimensionless variables

$$
\begin{gathered}
x^0 = Lx,\ y_* = L\delta y,\ z^0 = Lz_0 z,\ \rho^0 = \rho_\infty\tau^2\rho, \\
g^0 = U_\infty^2 g/2,\ p^0 = \rho_\infty U_\infty^2\tau^2 p,\ u^0 = U_\infty u,\ w^0 = U_\infty w, \\
v_* = U_\infty\delta z_0^{-1}v,\ \mu^0 = \mu_0\mu,\ F^0 = U_\infty\delta z_0^{-1}F, \\
\delta_w^0 = L\tau\delta_w,\ \delta_e^0 = L\delta\delta_e,\ \delta = \tau^{-1}z_0^{1/2}\mathrm{Re}^{-1/2},
\end{gathered}
\tag{1.1}
$$

where z_0 is the stretching which describes the ratio of the body dimensions in the transverse and longitudinal directions; τ, characteristic wing thickness; δ_e^0, boundary layer displacement thickness; $\mathrm{Re} = \rho_\infty U_\infty L/\mu_0$, Reynolds number; ρ_∞, gas density in the unperturbed stream; μ_0, viscosity, evaluated at the stagnation temperature in the incident flow; L, characteristic longitudinal dimension, which emerges from the final result in the similarity case; and g^0, stagnation enthalpy.

Substitution of the variables of Eq. (1.1) into the Navier–Stokes equations and going to the limit $\mathrm{Re} \to \infty$ leads to the equations of the three-dimensional boundary layer having the form, in Dorodnitsyn variables,

$$
\begin{gathered}
z_0 u\frac{\partial u}{\partial x} + v_0\frac{\partial u}{\partial\lambda} + w\frac{\partial u}{\partial z} = -\frac{z_0}{\rho}\frac{\partial p}{\partial x} + \frac{\partial}{\partial\lambda}\left(\rho\mu\frac{\partial u}{\partial\lambda}\right), \\
z_0 u\frac{\partial w}{\partial x} + v_0\frac{\partial w}{\partial\lambda} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial}{\partial\lambda}\left(\rho\mu\frac{\partial w}{\partial\lambda}\right), \\
z_0 u\frac{\partial g}{\partial x} + v_0\frac{\partial g}{\partial\lambda} + w\frac{\partial g}{\partial z} = \frac{\partial}{\partial\lambda}\left\{\rho\mu\left[\frac{1}{\sigma}\frac{\partial g}{\partial\lambda} - \frac{1-\sigma}{\sigma}\frac{\partial(u^2+w^2)}{\partial\lambda}\right]\right\}, \\
z_0\frac{\partial u}{\partial x} + \frac{\partial v_0}{\partial\lambda} + \frac{\partial w}{\partial z} = 0,\ \rho = \frac{2\gamma}{\gamma-1}\frac{p}{g-u^2-w^2},\ \mu = (g-u^2-w^2)^\omega, \\
\delta_e = \frac{\gamma-1}{2\gamma p}\int_0^\infty (g-u^2-w^2)\,d\lambda,
\end{gathered}
\tag{1.2}
$$

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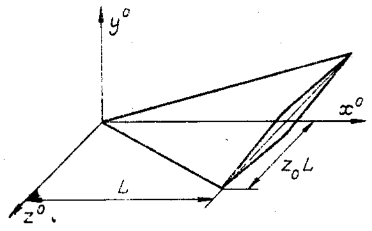

Fig. 1

$$\lambda = \int_0^y \rho dy, \quad v_0 = \rho v + w \frac{\partial \lambda}{\partial z} + z_0 u \frac{\partial \lambda}{\partial x}, \tag{1.2}$$

where σ is the Prandtl number. The boundary conditions take the form

$$u = w = 0, \ v_0 = \rho F, \ g = g_w \ (\lambda = 0), \ u \to 1, \ w \to 0, \ g \to 1 \ (\lambda \to \infty).$$

To evaluate the pressure we use the "tangent wedge" approximation [5] in a form valid for $M_\infty(\tau + \delta) \gg 1$,

$$p = \frac{\gamma+1}{2}\left(\frac{\partial \delta_w}{\partial x} + \chi \frac{\partial \delta_e}{\partial x}\right)^2, \tag{1.3}$$

where $\chi = \delta/\tau$ is the interaction parameter, describing the ratio of the boundary layer displacement thickness to the wing thickness.

We now consider flow over bodies of power-law shape $z_e = x^m$, $\delta_w = x^l \Delta_w(z/z_e)$, where z_e is the leading edge coordinate, and, following [6], we introduce the following variables:

$$\begin{gathered} x = x^*, \ z = x^m z^*, \ \lambda = x^k \lambda^*, \ g = g^*, \ \mu = \mu^*, \\ p = x^{2(l-1)} p^*, \ \rho = x^{2(l-1)} \rho^*, \ u = u^*, \ w = w^*, \ v_0 = x^n v^*, \\ \delta_e = x^{k-2(l-1)} \Delta_e, \ n = (2l - m - 2)/2, \ k = (2l + m - 2)/2. \end{gathered} \tag{1.4}$$

In using Eq. (1.3) to determine the pressure, with no mass transfer through the permeable surface, as was shown in [6], the interaction will be uniform over the body and the boundary-value problem of Eqs. (1.2) and (1.3) reduces to a similarity problem, if the parameters

$$m = 1, \ l = 3/4. \tag{1.5}$$

However, with forced blowing (or suction) through the body surface, to reduce the problem to a similarity one, one must also impose a restriction on the form of the function F so that v^* at the body surface should be independent of the coordinate x^*. In that case we obtain

$$F = x^{-1/4} F^*(z^*). \tag{1.6}$$

In the new variables of Eqs. (1.4) and (1.6), taking account of Eq. (1.5), the boundary-value problem of Eqs. (1.2) and (1.3) takes the form

$$\begin{gathered} (w^* - z_0 z^* u^*) \frac{\partial u^*}{\partial z^*} + \left(v^* - \frac{1}{4} z_0 \lambda^* u^*\right) \frac{\partial u^*}{\partial \lambda^*} = z_0 \frac{\gamma - 1}{2\gamma p^*} (g^* - u^{*2} - \\ - w^{*2}) \left(\frac{p^*}{2} + z^* \frac{dp^*}{dz^*}\right) + \frac{\partial}{\partial \lambda^*}\left(N^* \frac{\partial u^*}{\partial \lambda^*}\right), \\ (w^* - z_0 z^* u^*) \frac{\partial w^*}{\partial z^*} + \left(v^* - \frac{1}{4} z_0 \lambda^* u^*\right) \frac{\partial w^*}{\partial \lambda^*} = - \frac{\gamma - 1}{2\gamma p^*} (g^* - u^{*2} - w^{*2}) \frac{dp^*}{dz^*} + \frac{\partial}{\partial \lambda^*}\left(N^* \frac{\partial w^*}{\partial \lambda^*}\right), \\ (w^* - z_0 z^* u^*) \frac{\partial g^*}{\partial z^*} + \left(v^* - \frac{1}{4} z_0 \lambda^* u^*\right) \frac{\partial g^*}{\partial \lambda^*} = \frac{\partial}{\partial \lambda^*}\left\{N^* \left[\frac{1}{\sigma} \frac{\partial g^*}{\partial \lambda^*} - \frac{1 - \sigma}{\sigma} \frac{\partial (u^{*2} + w^{*2})}{\partial \lambda^*}\right]\right\}, \end{gathered} \tag{1.7}$$

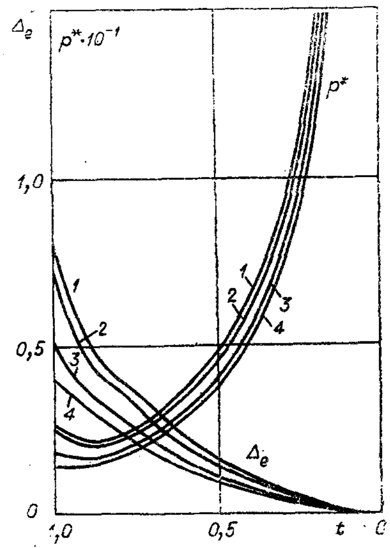

Fig. 2

$$\frac{\partial v^*}{\partial \lambda^*}+\frac{\partial w^*}{\partial z^*}-z_0\left(z^*\frac{\partial u^*}{\partial z^*}+\frac{\lambda^*}{4}\frac{\partial u^*}{\partial \lambda^*}\right)=0,$$

$$N^*=\frac{2\gamma p^*}{\gamma-1}(g^*-u^{*2}-w^{*2})^{\omega-1},\qquad \Delta_e=\frac{\gamma-1}{2\gamma p^*}\int\limits_0^\infty (g^*-u^{*2}-w^{*2})\,d\lambda^*,$$

$$p^*=\frac{\gamma+1}{2}\left[\frac{3}{4}\Delta_w-z^*\frac{d\Delta_w}{dz^*}+\chi\left(\frac{3}{4}\Delta_e-z^*\frac{d\Delta_e}{dz^*}\right)\right]^2. \tag{1.7}$$

The boundary conditions have the form

$$u^*=w^*=0,\ v^*=\rho^*F^*,\ g^*=g_w\ (\lambda^*=0),$$
$$u^*\to 1,\ w^*\to 0,\ g^*\to 1\ (\lambda^*\to\infty).$$

The system of equations obtained is independent of the coordinate x^* and describes flow in the three-dimensional boundary layer in the plane λ^*, z^*.

In what follows we assume that the body shape in the transverse section is given by the expression

$$\Delta_w=(1-z^{*2})^{\alpha_1}.$$

We note that for the parameter $\alpha_1 = 3/4$ at the edge of the body, we obtain conditions for moderate interaction [5].

2. To solve the system of equations (1.7), it is convenient to introduce the new variables [6], independent of the unknown functions

$$t=(1-z^*)^{2\alpha-1},\qquad \eta=\lambda\sqrt{\frac{(\gamma-1)(2\alpha-1)\,2^{2\alpha-3}}{\gamma(1-z^{*2})^{2\alpha-1}}}. \tag{2.1}$$

In addition, we must transform the components of the velocity vector in the boundary layer

$$w_0=-w^*,\qquad v^\lambda=\sqrt{\frac{(\gamma-1)(1-z^{*2})^{2\alpha-1}}{\gamma 2^{2\alpha-1}(2\alpha-1)}}\,\frac{v^*}{p^*}. \tag{2.2}$$

Substituting Eqs. (2.1) and (2.2) into the system of equations (1.7), we obtain

$$A\frac{\partial u^*}{\partial t}+B\frac{\partial u^*}{\partial \eta}=\frac{\partial}{\partial \eta}\left(N\frac{\partial u^*}{\partial \eta}\right)+z_0\frac{\gamma-1}{\gamma p^*}(g^*-u^{*2}-w_0^2)\times\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}\left(\frac{t}{2(2\alpha-1)}-\frac{1-t^{\frac{1}{2\alpha-1}}}{t^{\frac{3-4\alpha}{2\alpha-1}}p^*}\frac{dp^*}{dt}\right), \tag{2.3}$$

$$A\frac{\partial w_0}{\partial t}+B\frac{\partial w_0}{\partial\eta}=\frac{\partial}{\partial\eta}\left(N\frac{\partial w_0}{\partial\eta}\right)-\frac{\gamma-1}{\gamma}\left(g^*-u^{*2}-w_0^2\right)\times\frac{1}{t^{\frac{3-4\alpha}{2\alpha-1}}p^{*2}}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}\frac{dp^*}{dt},$$

$$A\frac{\partial g^*}{\partial t}+B\frac{\partial g^*}{\partial\eta}=\frac{\partial}{\partial\eta}\left\{N\left[\frac{1}{\sigma}\frac{\partial g^*}{\partial\eta}-\frac{1-\sigma}{\sigma}\frac{\partial\left(u^{*2}+w_0^2\right)}{\partial\eta}\right]\right\},$$

$$\frac{\partial v^1}{\partial\eta}=\frac{\eta\left(1-t^{\frac{1}{2\alpha-1}}\right)}{t^{\frac{2(1-\alpha)}{2\alpha-1}}p^*}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2(\alpha-1)}\left[\frac{\partial w_0}{\partial\eta}+z_0\left(1-t^{\frac{1}{2\alpha-1}}\right)\frac{\partial u^*}{\partial\eta}\right]-\frac{2}{t^{\frac{3-4\alpha}{2\alpha-1}}p^*}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}\left[\frac{\partial w_0}{\partial t}+z_0\left(1-t^{\frac{1}{2\alpha-1}}\right)\frac{\partial u^*}{\partial t}\right]$$

$$+z_0\eta\frac{t}{2(2\alpha-1)p^*}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}\frac{\partial u^*}{\partial\eta},$$

$$A=\left[w_0+z_0u^*\left(1-t^{\frac{1}{2\alpha-1}}\right)\right]\frac{2}{t^{\frac{3-4\alpha}{2\alpha-1}}p^*}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}, \tag{2.3}$$

$$B=v^1-\left[w_0+z_0u^*\left(1-t^{\frac{1}{2\alpha-1}}\right)\right]\frac{\eta\left(1-t^{\frac{1}{2\alpha-1}}\right)}{p^*t^{\frac{2(1-\alpha)}{2\alpha-1}}}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2(2\alpha-1)}-z_0u^*\eta\frac{t}{2(2\alpha-1)p^*}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1},$$

$$N=\left(g^*-u^{*2}-w_0^2\right)^{\alpha-1},$$

$$\Delta_e=\frac{1}{p^*}\sqrt{\frac{\gamma-1}{\gamma}\frac{t}{2\alpha-1}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}\int_0^\infty\left(g^*-u^{*2}-w_0^2\right)d\eta},$$

$$p^*=\frac{\gamma+1}{2}\left\{t^{\frac{\alpha_1-1}{2\alpha-1}}\left(2-t^{\frac{1}{2\alpha-1}}\right)^{\alpha_1-1}\left[\frac{3}{4}t^{\frac{1}{2\alpha-1}}\left(2-t^{\frac{1}{2\alpha-1}}\right)+2\alpha_1\left(1-t^{\frac{1}{2\alpha-1}}\right)^2\right]+\chi\left[\frac{3}{4}\Delta_e+\frac{2\alpha-1}{t^{\frac{2(1-\alpha)}{2\alpha-1}}}\left(1-t^{\frac{1}{2\alpha-1}}\right)\frac{d\Delta_e}{dt}\right]\right\}^2,$$

where the parameter $\alpha=\alpha_1$ for $\alpha_1\le 3/4$, and $\alpha=3/4$ for $\alpha_1\ge 3/4$. The boundary conditions have the form

$$u^*=w_0=0,\quad v^1|_w=\sqrt{\frac{\gamma-1}{\gamma}\frac{t}{2\alpha-1}\left(\frac{2-t^{\frac{1}{2\alpha-1}}}{2}\right)^{2\alpha-1}}\frac{\rho^*F^*}{p^*},$$

$$g^*=g_w(\eta=0),\quad u^*\to 1,\quad w_0\to 0,\quad g^*\to 1\quad(\eta\to\infty).$$

The system of boundary-layer equations, allowing for the viscous interaction, of Eq. (2.3), was solved by a relaxation method [7]. Depending on the sign of the coefficient $[w_0+z_0u^*(1-t^{1/2\alpha-1})]$, which determines the direction of the parabolic nature of Eq. (2.3), we used right-hand or left-hand derivatives with respect to the coordinate t.

To solve the system of equations (2.3), besides the boundary conditions at $\eta=0$ and ∞, we need conditions on the leading edges of the wing. Taking into account that in the vicinity of the edge $t=0$, for the displacement thickness we have an expansion $\Delta_e(t)=\Delta_{ek}t^{3/2}+\ldots$ [5], we can obtain an expression for the pressure

$$p^*|_{t\to 0}=\frac{\gamma+1}{2}\left\{2^{\alpha_1}\alpha_1 t^{\frac{\alpha_1-1}{2\alpha-1}}+\frac{3}{2}(2\alpha-1)\chi\Delta_{ek}t^{\frac{6\alpha-5}{2(2\alpha-1)}}\right\}^2. \tag{2.4}$$

Substituting Eq. (2.4) and $dp^*/dt|_{t\to 0}$ into the system of equations (2.3), and going to the limit $t\to 0$, for the flow around the leading edge of the wing we obtain a system of ordinary differential equations, which was solved by trial and error. It should be noted that near the leading edge the coefficient $[w_0+z_0u^*(1-t^{1/2\alpha-1})]$, which determines the direction of the parabolic feature, is positive for all values of η, and the flow in the boundary layer is directed from the leading edge to the wing axis.

In the present paper we consider symmetric flow over a wing, and there is therefore no need to calculate from one leading edge of the wing to the other, and it is enough to solve the system of equations from the edge to the wing plane of symmetry $(t=1)$ where the sink line occurs in this problem.

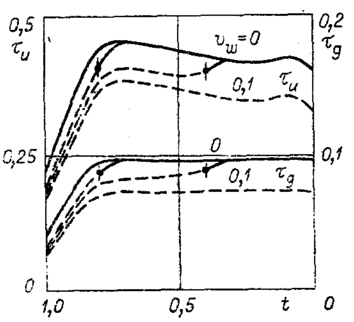

Fig. 3

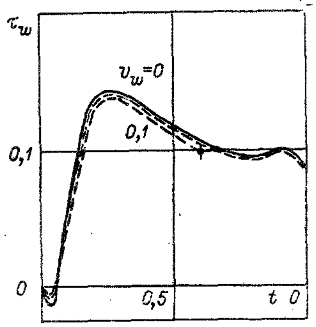

Fig. 4

To simplify the computations we assumed a linear dependence of viscosity on temperature $\omega = 1$, and also $\gamma = 1.4$, $\sigma = 0.71$. The investigations of the influence of mass transfer on the boundary layer characteristics carried out on a body with a sweepback angle of 45° ($z_0 = 1$), and a parameter governing the leading edge shape of $\alpha_1 = 3/4$. The body surface temperature was assumed to be constant, $g_w = 0.5$, and the interaction parameter, describing the ratio of the boundary layer displacement thickness to the wing thickness, $\chi = 1$.

Figure 2 shows the results of calculating the boundary layer displacement thickness Δ_e and the pressure p^* for constant values over the wing span of the function $v^1|_w = 0.2$; 0; -0.5; -1 (curves 1-4, respectively). The negative values of $v^1|_w$ correspond to suction of the boundary layer through the wing surface, and positive values correspond to blowing. Suction of the boundary layer, as one would expect, leads to an appreciable reduction in the boundary layer displacement thickness, which, in turn, leads to a reduced pressure. It is important to note that the pressure not only becomes less in magnitude, but also that the positive pressure gradient near the wing symmetry plane is considerably reduced. For the value $v^1|_w = -1$ it practically becomes equal to zero. Numerical calculations have shown that with no mass transfer in the boundary layer near the plane of symmetry there are reverse transverse flows which decrease with increase of the degree of suction, and vanish completely for $v^1|_w = -1$. But when there is blowing through the body surface, as can be seen from Fig. 2, near the symmetry plane there is an increase, both in the displacement thickness and in the pressure. The results of calculating the friction stresses in the longitudinal $\tau_u = \partial u^*/\partial\eta|_w$ and transverse $\tau_w = \partial w_0/\partial\eta|_w$ directions, and also the heat flux $\tau_g = \partial g^*/\partial\eta|_w$ for a wing with variable blowing over the wing span are given in Figs. 3 and 4.

The areas in which there is blowing of gas through the surface with $v^1|_w = 0.1$ correspond to the shaded parts of the curves. As can be seen from Fig. 3, variable blowing has an appreciable influence on the friction stress values τ_u and the heat flux τ_g, even at quite large distance from the point at which blowing begins, denoted by the vertical lines. For example, with blowing gas and $v^1|_w = 0.1$ in the region $0.4 \le t \le 1$ this influence appears up to values $t = 0.31$. Thus, the perturbation from blowing is propagated up the transverse flow to a distance $\Delta t = 0.09$. It is interesting to note that the length of this perturbed zone is practically independent of the coordinate where blowing begins ($t = 0.4$; 0.8). The influence of blowing on the friction stress in the transverse direction τ_w (Fig. 4) is quite weak, and the blowing leads to a reduced value of τ_w, which, apparently, is due to the increase in the boundary layer displacement thickness.

The results of investigating the influence of variable gas suction through the body surface on the boundary layer characteristics are shown in Figs. 5 and 6. The distribution of friction stress in the longitudinal direction $\tau_u = \partial u^*/\partial\eta|_w$ over the wing span with suction present is shown in Fig. 5. Areas in which there is suction of gas through the surface with values $v^1|_w = 0.1$ correspond to the broken parts of the curve. In the calculations the areas with gas suction start at the point $t = 0$; 0.2; 0.4; 0.8. As in the blowing case, the presence of variable suction has an appreciable influence on the characteristics of flow in the boundary layer, at a considerable distance from the point where blowing starts. Suction of the gas with values $v^1|_w = -1$ in the region $0.4 \le t \le 1$ begins to affect the functions τ_u and τ_g at the point $t = 0.28$, and, therefore, the perturbations are propagated up the transverse flow to a distance $\Delta t = 0.12$. It should be noted that there is a sharp increase in the heat fluxes and the friction stresses in the longitudinal direction near the beginning of the gas suction region. Figure 6 shows the results of calculated friction stresses in the transverse direction over the wing span $\tau_w = \partial w_0/\partial\eta|_w$. Near the beginning of the suction region there is an appreciable increase in the

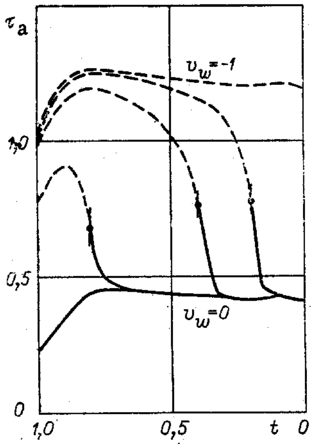

Fig. 5

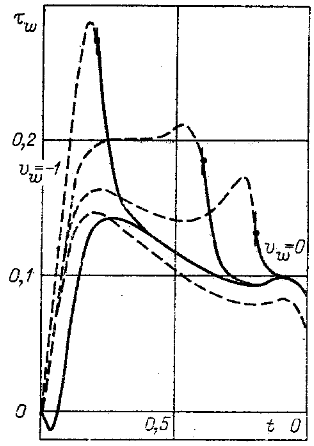

Fig. 6

friction stress τ_w. With suction of gas through the surface in the region $0.4 \leq t \leq 1$ the values of $0.3 \leq t \leq 0.47$ increase by more than a factor of τ_w. The increase in velocity of the transverse flow in the vicinity at which gas suction begins is explained by the fact that there is a change in the distribution of the boundary layer displacement thickness, and there is an increase in the pressure gradient in the transverse direction. It should be noted that when gas suction is present with values $v^1|_w = -1$ we have flow with a smooth inflow to the wing symmetry plane and no reverse flow regions are formed.

LITERATURE CITED

1. É. A. Stepanov, V. N. Charchenko, and Z. S. Ogorodnikova, "Gas flow with surface mass transfer," Review ONTI Tsentr. Aerodin. Gidrodin. Inst., No. 436 (1973).
2. A. V. Gomez, D. M. Curry, and C. G. Johnston, "Radiative, ablative and active cooling thermal protection studies for the leading edge of a fixed straight wing space shuttle," AIAA Paper N 71-445 (1971).
3. C. L. Scoville and P. D. Gorsuch, "Thermal protection for the space shuttle," Raumfahrforschung, Heft 2 (1971).
4. G. N. Dudin and V. Ya. Neiland, "Law of transverse flows for a three-dimensional boundary layer on a thin wing in hypersonic flow," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 2 (1976).
5. W. D. Hayes and R. F. Probstein (editors), Hypersonic Flow Theory, Academic Press (1967).
6. G. N. Dudin, "Interaction of hypersonic flow with the boundary layer on a thin triangular wing," Tr. Tsentr. Aerodin. Gidrodin. Inst., No. 1912 (1978).
7. G. E. Forsythe and W. R. Wasow, Finite Difference Methods for Partial Differential Equations, Wiley.